

ON BOOLEAN INTERVALS OF FINITE GROUPS

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ABSTRACT. We prove a dual version of a theorem of Øystein Ore, for any boolean interval of finite groups $[H, G]$ with a nonzero dual Euler totient $\hat{\varphi}$. For any boolean group-complemented interval, we observe that $\hat{\varphi} = \varphi \neq 0$ by the usual Ore's theorem. We conjecture that $\hat{\varphi}$ is always nonzero. We also discuss some applications in representation theory. In order to investigate the conjecture, we prove that for any boolean group-complemented interval, the graded coset poset $\hat{P} = \hat{C}(H, G)$ is Cohen-Macaulay and the non-trivial reduced Betti number of the order complex $\Delta(P)$ is $\hat{\varphi}$, so nonzero. We deduce that these results are true beyond the group-complemented case at index < 32 . One observes that they are also true when H is a Borel subgroup of G .

1. INTRODUCTION

The paper investigates a dual version of a theorem of Øystein Ore [12]. Although the main inspiration comes from the second author's work [13] on Ore's theorem for cyclic subfactor planar algebras, the paper is completely written in the framework of group theory, representation theory and combinatorics. Section 2 recalls some basic concepts and properties of distributive and boolean lattices. Section 3 first states the following Ore's theorem for which we give our own proof.

Theorem 1.1. *Let $[H, G]$ be a boolean interval of finite groups. Then $\exists g \in G$ such that $\langle Hg \rangle = G$.*

This result extends to any interval $[H, G]$ having a boolean top $[T, G]$ (i.e. T is the meet of the coatoms), like any finite distributive lattice.

Definition 1.2. *Let $[H, G]$ be a boolean interval of finite groups. Its Euler totient is defined by*

$$\varphi(H, G) := \sum_{K \in [H, G]} (-1)^{\ell(K, G)} |K : H|,$$

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where $\ell(K, G)$ is the length of the interval $[K, G]$.

We observe that for any boolean interval $[H, G]$, $\varphi(H, G)$ is exactly the number of cosets Hg such that $\langle Hg \rangle = G$, which is nonzero by Theorem 1.1.

Definition 1.3. Let $[H, G]$ be a boolean interval of finite groups. Its dual Euler totient is defined by

$$\hat{\varphi}(H, G) := \sum_{K \in [H, G]} (-1)^{\ell(H, K)} |G : K|.$$

We prove the following new basic result in finite group theory, which is almost a dual version of Ore's theorem for the boolean intervals.

Theorem 1.4. Let $[H, G]$ be a boolean interval of finite groups. If its dual Euler totient $\hat{\varphi}(H, G)$ is nonzero, then $\exists V$ an irreducible complex representation of G , such that $G_{(V^H)} = H$ (see Definition 3.4).

As for Theorem 1.1, Theorem 1.4 extends to any interval $[H, G]$ having a boolean bottom $[H, B]$ (i.e. B is the join of the atoms), like any finite distributive lattice. We deduce several applications of above theorem as a criterion (almost combinatorial) for a finite group G to be linearly primitive (i.e. existence of an irreducible faithful complex representation), and we find, in a purely combinatorial way, a non-trivial upper bound for the minimal number of irreducible complex representations generating (for \oplus and \otimes) the left regular representation of any finite group.

We observe that for any boolean interval $[H, G]$ satisfying $KK^{\complement} = G$ for any $\forall K \in [H, G]$ (with K^{\complement} the lattice-complement of K), the dual Euler totient $\hat{\varphi}(H, G) = \varphi(H, G)$, hence nonzero by Ore's theorem.

Conjecture 1.5. Let $[H, G]$ be boolean. Then $\hat{\varphi}(H, G) \neq 0$.
[It is checked by GAP [8] for $|G : H| < 32$.]

If the conjecture holds, then the statement of Theorem 1.4 is true without assuming $\hat{\varphi}$ nonzero; this would solve a conjecture of [13].

Section 4 exposes the first results we get by investigating Conjecture 1.5. For any interval $[H, G]$, we recall the proof that the Möbius invariant of its bounded coset poset is

$$\mu(\hat{C}(H, G)) = - \sum_{K \in [H, G]} \mu(K, G) |G : K|$$

and we observe (after Russ Woodroffe) that in the rank $n + 1$ boolean case, $\mu(\hat{C}(H, G))$ is exactly $(-1)^n \hat{\varphi}(H, G)$. So we are reduced to investigate this Möbius invariant. Thus Conjecture 1.5 is equivalent to

say that $\mu(\hat{C}(H, G)) \neq 0$ for any boolean interval $[H, G]$. As explained in [15], the non-vanishing of $\mu(\hat{C}(1, G))$ is conjectured by Brown for any finite group G . A weaker version, namely, $\Delta(\hat{C}(1, G))$ is not contractible, has been proved in [15]. This leads to the following weaker version of our Conjecture 1.5:

Conjecture 1.6. *For $[H, G]$ boolean, $\Delta(\hat{C}(H, G))$ is not contractible.*

The Möbius invariant is well-known to be the (reduced) Euler characteristic of the order complex $\Delta(P)$:

$$\tilde{\chi}(\Delta(P)) = \sum_{k=-1}^n (-1)^k \tilde{\beta}_k(\Delta(P))$$

with $\dim(\Delta(P)) = n$ and $\tilde{\beta}_k(\Delta(P))$ the k th reduced Betti number, i.e. the dimension of k th reduced homology space $\tilde{H}_k(\Delta(P))$. We recall that if $\hat{P} = \hat{C}(H, G)$ is Cohen-Macaulay, then by definition $\forall k < n$, $\tilde{\beta}_k(\Delta(P)) = 0$, so that $\tilde{\beta}_n(\Delta(P)) = (-1)^n \mu(\hat{P}) = \hat{\varphi}(H, G)$. A sufficient condition for a graded poset to be Cohen-Macaulay is the existence of a (dual) EL-labeling [1]. Russ Woodroffe suggested a labeling for $\hat{C}(H, G)$ with $[H, G]$ boolean. We prove that it is a dual EL-labeling iff $[H, G]$ is also group-complemented, which leads to:

Theorem 1.7. *Let $[H, G]$ be a boolean group-complemented interval of rank $n + 1$. Then $\hat{C}(H, G)$ is Cohen-Macaulay; moreover $\tilde{\beta}_n(\Delta(P)) = \hat{\varphi}(H, G) \neq 0$; it is the number of cosets Hg generating G individually.*

At index $|G : H| < 32$, there are 612 boolean intervals (up to equivalence), they are all group-complemented, except $[D_8, A_2(2)]$ and $[S_3, A_2(2)]$, both of rank 2, so their graded coset posets are also Cohen-Macaulay and $\hat{\varphi}$ is nonzero. It follows that:

Corollary 1.8. *For any boolean interval $[H, G]$ of index $|G : H| < 32$, the graded coset poset $\hat{P} = \hat{C}(H, G)$ is Cohen-Macaulay; moreover the nontrivial reduced Betti number of $\Delta(P)$ is nonzero.*

Question 1.9. *Can Corollary 1.8 be extended to any boolean interval?*

If G has a BN-pair (as any finite simple group of Lie type) of rank n with B being the corresponding Borel subgroup, then $[B, G]$ is boolean of rank n . One observes that the above question has a positive answer for such intervals.

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2. BASICS IN LATTICE THEORY

A lattice (L, \wedge, \vee) is a poset L in which every two elements a, b have a unique supremum (or join) $a \vee b$ and a unique infimum (or meet) $a \wedge b$. Let G be a finite group. The set of subgroups $K \subseteq G$ forms a lattice, denoted by $\mathcal{L}(G)$, ordered by \subseteq , with $K_1 \vee K_2 = \langle K_1, K_2 \rangle$ and $K_1 \wedge K_2 = K_1 \cap K_2$. A sublattice of (L, \wedge, \vee) is a subset $L' \subseteq L$ such that (L', \wedge, \vee) is also a lattice. Let $a, b \in L$ with $a \leq b$, then the interval $[a, b]$ is the sublattice $\{c \in L \mid a \leq c \leq b\}$. Any finite lattice admits a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$. Atoms (resp. coatoms) are minimum (resp. maximum) elements in $L \setminus \{\hat{0}\}$ (resp. $L \setminus \{\hat{1}\}$). The top (resp. bottom) interval of a finite lattice L is the interval $[t, \hat{1}]$ with t the meet of all the coatoms (resp. $[\hat{0}, b]$ with b the join of all the atoms). The length of a finite lattice L is the greatest length of a maximal chain. A lattice is distributive if the join and meet operations distribute over each other. A distributive lattice is called boolean if any element b admits a unique complement b^c (i.e. $b \wedge b^c = \hat{0}$ and $b \vee b^c = \hat{1}$). The subset lattice of $\{1, 2, \dots, n\}$, with union and intersection, is called the boolean lattice B_n of rank n . Any finite boolean lattice is isomorphic to some B_n .

Lemma 2.1. *The top and bottom intervals of a distributive lattice are boolean.*

Proof. See [16, items a-i p254-255]. It follows from Birkhoff's representation theorem (a finite lattice is distributive iff it embeds into a finite boolean lattice). \square

We refer to [16] for more details on lattice basics.

3. ORE'S THEOREM AND DUAL VERSION

3.1. Ore's theorem on boolean intervals of finite groups.

Øystein Ore has proved the following result in [12, Theorem 4, p267].

Theorem 3.1. *A finite group G is cyclic if and only if its subgroup lattice $\mathcal{L}(G)$ is distributive.*

Proof. (\Leftarrow): It is just a particular case of Theorem 3.3 with $H = \{1\}$. (\Rightarrow): A finite cyclic group $G = \mathbb{Z}/n$ has exactly one subgroup of order d , denoted by \mathbb{Z}/d , for every divisor d of $|G|$. Now $\mathbb{Z}/d_1 \vee \mathbb{Z}/d_2 = \mathbb{Z}/\text{lcm}(d_1, d_2)$ and $\mathbb{Z}/d_1 \wedge \mathbb{Z}/d_2 = \mathbb{Z}/\text{gcd}(d_1, d_2)$, but lcm and gcd are distributive, so the result follows. \square

Definition 3.2. *An interval of finite groups $[H, G]$ is called w-cyclic (or weakly cyclic) if there is $g \in G$ such that $\langle H, g \rangle = G$. Note that $\langle H, g \rangle = \langle Hg \rangle$.*

Øystein Ore has extended one side of Theorem 3.1 to the interval of finite groups [12, Theorem 7] for which we will give our own proof:

Theorem 3.3. *A distributive interval $[H, G]$ is w-cyclic.*

Proof. The proof follows from the last two claims and Lemma 2.1.

Claim: Let G be a finite group and M a maximal subgroup. Then $[M, G]$ is w-cyclic.

Proof: Let $g \in G$ such that $g \notin M$, then $\langle M, g \rangle = G$ by maximality. \blacksquare

Claim: A boolean interval $[H, G]$ is w-cyclic.

Proof: Let M be a coatom of $[H, G]$, and M^c be its complement. By the previous claim and induction on the length of the lattice, we can assume $[H, M]$ and $[H, M^c]$ both to be w-cyclic, i.e. there are $a, b \in G$ such that $\langle H, a \rangle = M$ and $\langle H, b \rangle = M^c$. For $g = ab$, $a = gb^{-1}$ and $b = a^{-1}g$, so $\langle H, a, g \rangle = \langle H, g, b \rangle = \langle H, a, b \rangle = M \vee M^c = G$. Now, $\langle H, g \rangle = \langle H, g \rangle \vee H = \langle H, g \rangle \vee (M \wedge M^c)$ but by distributivity $\langle H, g \rangle \vee (M \wedge M^c) = (\langle H, g \rangle \vee M) \wedge (\langle H, g \rangle \vee M^c)$. So $\langle H, g \rangle = \langle H, a, g \rangle \wedge \langle H, g, b \rangle = G$. The result follows. \blacksquare

Claim: An interval $[H, G]$ is w-cyclic if its top interval is so.

Proof: Let $[K, G]$ be the top interval and $g \in G$ with $\langle K, g \rangle = G$. For any maximal subgroup M of G , we have $K \subseteq M$ by definition, and so $g \notin M$, then a fortiori $\langle H, g \rangle \not\subseteq M$. It follows that $\langle H, g \rangle = G$. \blacksquare

\square

3.2. A dual version of Ore's theorem.

In this section, we extend a result on intervals of finite groups $[H, G]$, obtained in a previous paper of the first author [13, corollary 1.9(1)].

Definition 3.4. Let W be a representation of a group G , K a subgroup of G , and X a subspace of W . We define the fixed-point subspace

$$W^K := \{w \in W \mid kw = w, \forall k \in K\}$$

and the pointwise stabilizer subgroup

$$G_{(X)} := \{g \in G \mid gx = x, \forall x \in X\}.$$

Remark 3.5. Note that

- $K_1 \subseteq K_2 \Rightarrow W^{K_2} \subseteq W^{K_1}$.
- $X_1 \subseteq X_2 \Rightarrow G_{(X_2)} \subseteq G_{(X_1)}$.

Lemma 3.6. Let G be a finite group, $K, K' \in \mathcal{L}(G)$ and V be a representation of G . Then $V^{K \vee K'} = V^K \cap V^{K'}$.

Proof. First $K, K' \subseteq K \vee K'$, so $V^{K \vee K'}$ is included in V^K and $V^{K'}$, so in $V^K \cap V^{K'}$. Now let $v \in V^K \cap V^{K'}$, then $\forall k \in K$ and $\forall k' \in K'$, $kv = k'v = v$, but any element $g \in K \vee K'$ is of the form $k_1 k'_1 k_2 k'_2 \cdots k_r k'_r$ with $k_i \in K$ and $k'_i \in K'$, it follows that $gv = v$ and so $V^K \cap V^{K'} \subseteq V^{K \vee K'}$. \square

Lemma 3.7. If G is a finite group, H a subgroup and V a representation of G , then

- (1) $H \subseteq G_{(V^H)}$.
- (2) $V^{G_{(V^H)}} = V^H$.

Proof. Let $h \in H$ and $v \in V^H$. Then by definition $hv = v$, so $H \subseteq G_{(V^H)}$; and therefore $V^{G_{(V^H)}} \subseteq V^H$. Now let $v \in V^H$ and $g \in G_{(V^H)}$, then by definition $gv = v$, it follows that $V^H \subseteq V^{G_{(V^H)}}$ also. \square

Definition 3.8. An interval of finite groups $[H, G]$ is called linearly primitive if there is an irreducible complex representation V of G such that $G_{(V^H)} = H$.

Remark 3.9. For $H = \{e\}$, we recover the usual linear primitivity, i.e. the existence of an irreducible faithful complex representation.

Lemma 3.10. Let $H \subseteq K \subseteq G$ be a chain of finite groups and V be a representation of G , then

$$V^K \subsetneq V^H \Rightarrow K \not\subseteq G_{(V^H)}.$$

Proof. Suppose that $K \subseteq G_{(V^H)}$, then $V^K \supseteq V^{G_{(V^H)}} = V^H$ by Lemma 3.7. Hence $V^K = V^H$, contradiction with $V^K \subsetneq V^H$. \square

Let $[\cdot, \cdot]_G$ be the usual normalized inner product of finite dimensional complex representations of a finite group G .

Lemma 3.11 (Frobenius reciprocity, [9] p62). *Let G be a finite group and H a subgroup. Let V (resp. W) be a finite dimensional complex representation of G (resp. of H). Let $\text{Ind}(W)$ be the induction to G and $\text{Res}(V)$ the restriction to H , then $[V, \text{Ind}(W)]_G = [\text{Res}(V), W]_H$.*

Lemma 3.12. *Let $[H, G]$ be an interval of finite groups. Let V_1, \dots, V_r be the irreducible complex representations of G (up to equivalence). Then*

$$|G : H| = \sum_i \dim(V_i) \dim(V_i^H).$$

Proof. The following proof is suggested by Tobias Kildetoft. Let 1_H^G be the trivial representation of H induced to G . On one hand, it has dimension $|G : H|$, and on the other hand, this dimension is also

$$\sum_i \dim(V_i) [V_i, 1_H^G]_G = \sum_i \dim(V_i) [V_i, 1_H]_H = \sum_i \dim(V_i) \dim(V_i^H).$$

The first equality follows from Frobenius reciprocity. \square

Definition 3.13. *Let $[H, G]$ be a boolean interval of finite groups. We define its Euler totient by*

$$\varphi(H, G) := \sum_{K \in [H, G]} (-1)^{\ell(K, G)} |K : H|$$

and its dual Euler totient by

$$\hat{\varphi}(H, G) := \sum_{K \in [H, G]} (-1)^{\ell(H, K)} |G : K|.$$

Remark 3.14. *In general $\varphi(H, G) \neq \hat{\varphi}(H, G)$. The smallest example is $[D_8, A_2(2)]$ with $\varphi = 16$ and $\hat{\varphi} = 8$. Here $A_2(2)$ denotes the simple group of order 168, and D_8 , the dihedral group of order 8.*

Lemma 3.15. *For a boolean interval $[H, G]$, $\varphi(H, G) = |E|/|H|$ with $E = \{g \in G \mid \langle H, g \rangle = G\}$. Then $\varphi(H, G)$ is exactly the number of cosets Hg generating G individually.*

Proof. Let M_1, \dots, M_n be the coatoms of $[H, G]$. Then

$$-|\bigcup_i M_i| = \sum_{t=1}^n (-1)^t \sum_{i_1 < \dots < i_t} |M_{i_1} \cap \dots \cap M_{i_t}|$$

by inclusion-exclusion principle; but $g \in G \setminus \bigcup_i M_i$ iff $g \in E$. The result follows by the boolean structure of $[H, G]$. \square

Remark 3.16. By Lemma 2.1, the function φ can be extended to any distributive interval as

$$\varphi(H, G) = |T : H| \cdot \varphi(T, G)$$

with $[T, G]$ the top interval of $[H, G]$, so that for $n = \prod_i p_i^{n_i}$,

$$\varphi(1, \mathbb{Z}/n) = \prod_i p_i^{n_i-1} \cdot \prod_i (p_i - 1)$$

which is the usual Euler totient $\varphi(n)$.

Proposition 3.17. For $[H, G]$ boolean, $\varphi(H, G) > 0$.

Proof. By Theorem 3.3, $\exists g \in G$ with $\langle H, g \rangle = G$, so $g \notin M_i \forall i$, thus $G \setminus \cup_i M_i \neq \emptyset$, and so $\varphi(H, G) > 0$ by Lemma 3.15. \square

Let $[H, G]$ be boolean of rank $n + 1$.

Question 3.18. Is it true that $\varphi(H, G)$ and $\hat{\varphi}(H, G) \geq 2^n$?

This is checked by GAP for $|G : H| < 32$. See also Example 4.37.

If this lower bound is correct, then it is optimal because it is realized by $[1 \times S_2^n, S_2 \times S_3^n]$.

Conjecture 3.19. If $[H, G]$ is boolean, then $\hat{\varphi}(H, G) \neq 0$.

Remark 3.20. For $[H, G]$ boolean with $|G : H|$ a prime-power p^m ,

$$\hat{\varphi}(H, G) = \sum_{K \in [H, G]} (-1)^{\ell(H, K)} |G : K| + (-1)^{\ell(H, G)} \neq 0$$

because the first component of the above sum is a multiple of p .

The following is the main theorem of Section 3. It is a dual version of Theorem 3.1 and a new basic result in finite group theory.

Theorem 3.21. Let $[H, G]$ be a boolean interval of finite groups. If $\hat{\varphi}(H, G) \neq 0$, then $[H, G]$ is linearly primitive.

Proof. We will use the notations of Lemma 3.12. Consider the sum

$$S(i) := \sum_{K \in [H, G]} (-1)^{\ell(H, K)} \dim(V_i^K).$$

Let K_1, \dots, K_n be the atoms of $[H, G]$. Let A_i be the set of atoms K satisfying $V_i^K = V_i^H$, and B_i the set of atoms not in A_i . Let K_{A_i} (resp. K_{B_i}) be the join of all the elements of A_i (resp. B_i).

Claim: For $K \in [H, G]$, $V_i^K = V_i^H \Leftrightarrow K \in [H, K_{A_i}]$.

Proof: By the boolean structure, for each $K \in [H, G]$, $\exists J \subseteq \{1, 2, \dots, n\}$ such that $K = \vee_{j \in J} K_j$ and by Lemma 3.6, $V_i^{\vee_{j \in J} K_j} = \bigcap_{j \in J} V_i^{K_j}$, so $V_i^K = V_i^H$ if and only if $V_i^{K_j} = V_i^H \forall j \in J$. \blacksquare

Again, by the boolean structure, we have

$$[H, G] = [H, K_{A_i}] \vee [H, K_{B_i}] = \bigsqcup_{K \in [H, K_{A_i}]} K \vee [H, K_{B_i}],$$

so with

$$T(i) := \sum_{K \in [H, K_{B_i}]} (-1)^{\ell(H, K)} \dim(V_i^K)$$

and Lemma 3.6, we get that

$$S(i) = \sum_{K \in [H, K_{A_i}]} (-1)^{\ell(H, K)} T(i) = T(i) \cdot (1 - 1)^{|A_i|}.$$

Claim: A boolean interval $[H, G]$ is linearly primitive if and only $\exists i$ with $|A_i| = 0$.

Proof: First if $\exists i$ such that $|A_i| = 0$, then for any atom K , $V_i^K \subsetneq V_i^H$, and so by Lemma 3.10, $K \not\subseteq G_{(V_i^H)}$. In other words, $G_{(V_i^H)}$ contains no atom, but it contains H , so $G_{(V_i^H)} = H$.

Next if $\exists i$ such that $G_{(V_i^H)} = H$, then any atom satisfies $K \not\subseteq G_{(V_i^H)}$. But if $|A_i| \neq 0$, then there is an atom K with $V_i^K = V_i^H$, so $G_{(V_i^H)} = G_{(V_i^K)} \supset K$ (by Lemma 3.7), contradiction with $K \not\subseteq G_{(V_i^H)}$. ■

Hence if $[H, G]$ is not linearly primitive, then $\forall i$ $|A_i| \neq 0$, and so $S(i) = 0$; but by Lemma 3.12, $\hat{\varphi}(H, G) = \sum_{i=1}^r \dim(V_i) S(i) = 0$; the result follows. □

Conversely,

Question 3.22. *Can we deduce that $\hat{\varphi}(H, G) \neq 0$ assuming that $[H, G]$ is boolean and linearly primitive?*

Lemma 3.23. *Let $[H, G]$ be a boolean interval of finite groups and let $K \in [H, G]$. The following are equivalent:*

- (1) $KK^{\mathbb{C}} = K^{\mathbb{C}}K$
- (2) $KK^{\mathbb{C}} = G$
- (3) $|G : K| = |K^{\mathbb{C}} : H|$

with $K^{\mathbb{C}}$ the lattice-complement of K in $[H, G]$.

Proof. (1) \Rightarrow (2): By definition of the lattice-complement, $K \vee K^{\mathbb{C}} = G$ (and $K \wedge K^{\mathbb{C}} = H$), but any element in $K \vee K^{\mathbb{C}}$ is of the form $k_1 k'_1 k_2 k'_2 \cdots k_r k'_r$ with $k_i \in K$ and $k'_i \in K^{\mathbb{C}}$, so by (1) any such element is of the form kk' with $k \in K$ and $k' \in K^{\mathbb{C}}$, i.e. $G = KK^{\mathbb{C}}$.

(1) \Leftarrow (2): Immediate.

(2) \Leftrightarrow (3): By the product formula, $|KK^{\mathbb{C}}| = |K||K^{\mathbb{C}}|/|H|$, so $KK^{\mathbb{C}} = G$ iff $|KK^{\mathbb{C}}| = |G|$, iff $|G : K| = |G|/|K| = |K^{\mathbb{C}}|/|H| = |K^{\mathbb{C}} : H|$. □

Definition 3.24. A boolean interval $[H, G]$ is called group-complemented if every $K \in [H, G]$ satisfies one of the equivalent statements of Lemma 3.23.

Remark 3.25. There are boolean intervals of finite groups which are not group-complemented, the smallest example is $[D_8, A_2(2)]$ of index 21 and rank 2. It is not group-complemented because by GAP, there is $K \in [D_8, A_2(2)]$ with $|D_8 : K| = 7 \neq 3 = |K^\complement : A_2(2)|$. At index < 32 , there is only one other example given by $[S_3, A_2(2)]$.

Definition 3.26. The interval $[H, G]$ is called Dedekind if every $K \in [H, G]$ and every $g \in G$ satisfy $HgK = KgH$.

Lemma 3.27. For a boolean interval of finite groups, Dedekind implies group-complemented.

Proof. By assumption, for every $K \in [H, G]$ and every $g \in G$, we have $HgK = KgH$. It follows that $HK^\complement K = KK^\complement H$, but $HK^\complement = K^\complement H = K^\complement$, so $KK^\complement = K^\complement K$. \square

Remark 3.28. The converse of Lemma 3.27 is not true. There are boolean group-complemented intervals which are not Dedekind, for example $[H, G]$ as follows:

gap> G:=TransitiveGroup(d,r); H:=Stabilizer(G,1);
with $(d, r) = (10, 4)$ or $(30, 7)$.

Lemma 3.29. For a boolean group-complemented interval $[H, G]$, $\hat{\varphi}(H, G) = \varphi(H, G)$.

Proof. By definition

$$\varphi(H, G) = \sum_{K \in [H, G]} (-1)^{\ell(K, G)} |K : H|$$

but $\ell(K, G) = \ell(H, K^\complement)$; moreover by the group-complemented assumption and Lemma 3.23 we have $|K : H| = |G : K^\complement|$, so

$$\varphi(H, G) = \sum_{K \in [H, G]} (-1)^{\ell(H, K^\complement)} |G : K^\complement|.$$

The result follows by the change of variable $K \leftrightarrow K^\complement$. \square

Corollary 3.30. Let $[H, G]$ be a boolean interval. If it is group-complemented, then $\hat{\varphi}(H, G) > 0$.

Proof. By Lemma 3.29 and Proposition 3.17. \square

Remark 3.31. The converse of Corollary 3.30 is false because $[D_8, A_2(2)]$ is boolean and not group-complemented but its $\hat{\varphi} = 8 > 0$.

Corollary 3.32. *If a boolean interval of finite groups $[H, G]$ is group-complemented, then it is linearly primitive.*

Proof. By Corollary 3.30, $\hat{\varphi}(H, G) \neq 0$, so we apply Theorem 3.21. \square

By Remark 3.28, Corollary 3.32 is strictly stronger than a result in a previous paper (where *group-complemented* is replaced by *Dedekind* [13, Corollary 1.9 (1)]). Moreover by Remark 3.31, Theorem 3.21 is strictly stronger than Corollary 3.32.

3.3. Applications to representation theory. We get a criterion for a finite group to be linearly primitive. We also deduce a non-trivial upper bound for the minimal number of irreducible complex representations generating the left regular representation.

Lemma 3.33. *An interval $[H, G]$ is linearly primitive if its bottom interval is so.*

Proof. Let $[H, K]$ be the bottom interval, i.e. $K = \bigvee_i K_i$ with K_1, \dots, K_n the minimal overgroups of H . By assumption, there is an irreducible complex representation W of K such that $K_{(W^H)} = H$. Let V be an irreducible complex representation of G such that its restriction on K admits W as subrepresentation. Now $W \subseteq V$, so that $W^H \subseteq V^H$ and hence

$$H \subseteq K_{(V^H)} \subseteq K_{(W^H)} = H.$$

It follows that $K_{(V^H)} = H$. Now if $\exists i$ such that $V^H = V^{K_i}$, then

$$K_i \subseteq K_{(V^{K_i})} = K_{V^H},$$

contradiction with $H \subsetneq K_i$. So $\forall i$ $V^{K_i} \subsetneq V^H$. Then by Lemma 3.10, $K_i \not\subseteq G_{(V^H)} \forall i$, so by minimality $G_{(V^H)} = H$. \square

Corollary 3.34. *If the interval $[H, G]$ admits a boolean bottom interval $[H, K]$ with $\hat{\varphi}(H, K) \neq 0$, then it is linearly primitive.*

Proof. By Theorem 3.21 and Lemma 3.33. \square

Definition 3.35. *Let G be a group. A subgroup H is called core-free if $\forall N \triangleleft G$, then $N \subset H$ implies $N = \{1\}$.*

The following theorem is almost a combinatorial criterion for a finite group to be linearly primitive.

Theorem 3.36. *Let G be a finite group. If G admits a core-free subgroup H such that the bottom interval $[H, K]$ of $[H, G]$ is boolean with a nonzero dual Euler totient, then G is linearly primitive.*

Proof. By Corollary 3.34, the interval $[H, G]$ is linearly primitive. So there is an irreducible complex representation V with $G_{(V^H)} = H$. Now, $V^H \subseteq V$ so $G_{(V)} \subseteq G_{(V^H)}$, but $\ker(\pi_V) = G_{(V)}$, it follows that $\ker(\pi_V) \subseteq H$; but H is a core-free subgroup of G , and $\ker(\pi_V)$ a normal subgroup of G , so $\ker(\pi_V) = \{e\}$, which means that V is faithful on G , i.e. G is linearly primitive. \square

There is the following trivial consequence of Theorem 3.36.

Corollary 3.37. *If a finite group G admits a core-free maximal subgroup, then it is linearly primitive.*

Proof. Let M be a maximal subgroup of G , then $[M, G]$ is boolean (so equal to its bottom interval) of rank 1 (so $\hat{\varphi}(M, G) \neq 0$). Hence, by Theorem 3.36, if M is core-free, then G is linearly primitive. \square

We reformulate Theorem 3.36 for p -groups as follows:

Corollary 3.38. *Let G be a finite p -group with a core-free subgroup H such that $N_G(H)/H$ cyclic or generalized quaternion. Then G has a cyclic center.*

Proof. A cyclic or generalized quaternion p -group has a unique subgroup of order p . So $[H, N_G(H)]$ has a unique atom B' . Since a maximal subgroup of a p -group is normal [20, Corollary 1 p137], H is normal in any atom of $[H, G]$, so is in B (the join of all the atoms). It follows that $B \subseteq N_G(H)$ the normalizer of H in G , and $B = B'$. It follows that $[H, G]$ has a boolean bottom, but by Remark 3.20, $\hat{\varphi}(H, G) \neq 0$, so by Theorem 3.36, G is linearly primitive, which (for a p -group) is equivalent to have a cyclic center [9, Theorem 2.32]. \square

We consider the following converse to Theorem 3.36.

Question 3.39. *Does a linearly primitive group G admit a core-free subgroup H such that the bottom interval $[H, B]$ of $[H, G]$ is boolean with a nonzero dual Euler totient?*

A positive answer to the above question leads to the following:

Statement 3.40. *A finite p -group G with a cyclic center has a core-free subgroup H with $N_G(H)/H$ cyclic or generalized quaternion.*

Proof. Let G be a finite p -group with a cyclic center. Then G is linearly primitive, so there is a core-free subgroup H such that the bottom interval $[H, B]$ of $[H, G]$ is boolean. Then $B \subseteq N_G(H)$, as for Corollary 3.38. But $[H, B] \simeq [1, B/H]$ as lattices, so the subgroup lattice $\mathcal{L}(B/H)$ is boolean, and by Ore's Theorem 3.1, B/H is cyclic. Now it is also a p -group, so $B/H \simeq \mathbb{Z}/p^m$. But $\mathcal{L}(B/H)$ is boolean, so

$m = 1$. It follows that B is the unique atom of $[H, G]$, and B/H is the unique atom of $\mathcal{L}(N_G(H)/H)$. So $N_G(H)/H$ is a p -group with a unique subgroup of order p , hence it is cyclic or generalized quaternion [20, Theorem 15]. \square

Remark 3.41. *By using the proof of Corollary 3.38, the above statement is in fact equivalent to a positive answer of Question 3.39 for any p -group G , which then is true if $|G| < 512$, by GAP computation. Moreover, it is true for $|G| < 2187$, if $p > 2$.*

Definition 3.42. *Let $[H, G]$ be an interval of finite groups. The bottom boolean Euler length $bbel(H, G)$ is the minimal length for an ordered chain of subgroups*

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that the interval $[H_{\alpha-1}, H_\alpha]$ admits a boolean bottom with a nonzero dual Euler totient.

Theorem 3.43. *For any interval $[H, G]$, the minimal cardinal for a set $\{V_i \mid i \in I\}$ of irreducible complex representations of G such that $\bigcap_{i \in I} G_{(V_i^H)} = H$, is less than $bbel(H, G)$.*

Proof. Consider a chain as in Definition 3.42, and of length $bbel(H, G)$. By Corollary 3.34, for any α there exists an irreducible complex representation W_α of H_α such that $T_{(W^S)} = S$ with $T = H_\alpha$, $S = H_{\alpha-1}$ and $W = W_\alpha$. Let V_α be an irreducible complex representation of G such that its restriction to H_α admits W_α as subrepresentation. It follows that $V_1, \dots, V_{bbel(H, G)}$ satisfy $\bigcap_{\alpha=1}^{bbel(H, G)} G_{(V_\alpha^H)} = H$. \square

Definition 3.44. *Let G be a finite group. The length $bbel(G)$ is defined as $bbel(\{e\}, G)$. It is a purely combinatorial invariant of G .*

Corollary 3.45. *For any finite group G , the minimal number of irreducible complex representations generating the left regular representation of G (for \oplus and \otimes) is bounded above by $bbel(G)$.*

Proof. By Theorem 3.43, we can find irreducible complex representations $V_1, \dots, V_{bbel(G)}$ satisfying $\bigcap_{\alpha=1}^{bbel(G)} G_{(V_\alpha)} = \{e\}$, but $G_{(V_\alpha)} = \ker(\pi_\alpha)$. So $\bigcap_{\alpha=1}^{bbel(G)} \ker(\pi_\alpha) = \{e\}$, which implies that $V_1, \dots, V_{bbel(G)}$ generate the left regular representation of G (for \oplus and \otimes). \square

Note that, by Theorem 3.36, the following is a better upper bound,

$$cfel(G) := \min\{bbel(H, G) \mid H \text{ core-free}\},$$

but it requires to know all the normal subgroups of G .

4. COHEN-MACAULAY COSET POSET

All the posets are assumed to be finite. A poset P is bounded if it admits a smallest element $\hat{0}$ and a greatest element $\hat{1}$. Given a poset P , the bounded extension of P is defined as $\hat{P} := P \sqcup \{\hat{0}, \hat{1}\}$. The proper part of $Q := \hat{P}$ is defined as $\bar{Q} := P$.

4.1. Order complex. An abstract simplicial complex Δ on a finite set V is a nonempty collection of subsets of V such that $E \in \Delta$ and $F \subseteq E$ implies $F \in \Delta$. The elements of V are called vertices and the elements of Δ are called faces of the simplicial complex Δ . We define the dimension of a face F as $\dim F := |F| - 1$. Faces of dimension d are referred to as d -faces. The dimension of the complex $\dim(\Delta)$ is defined as the largest dimension of any of its faces.

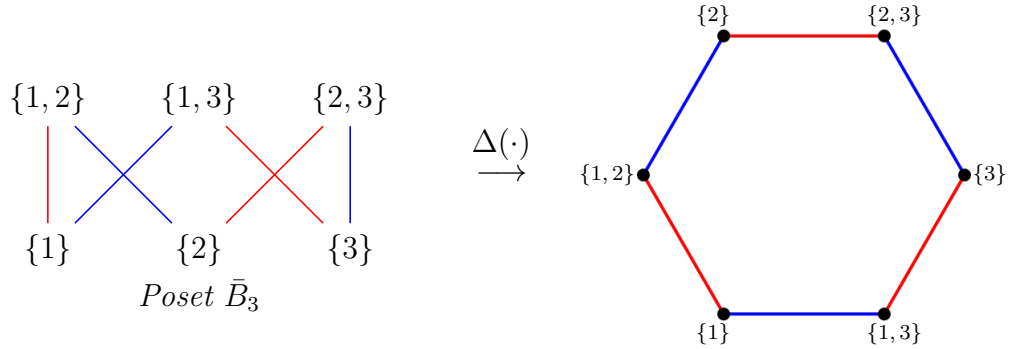
Let P be a poset. One can associate to P an abstract simplicial complex, $\Delta(P)$ which is defined as follows: The vertices of $\Delta(P)$ are the elements of P and the faces of $\Delta(P)$ are the chains (i.e., totally ordered subsets) of P ; $\Delta(P)$ is called the order complex of P . Any topological property attributed to $\Delta(P)$ will be considered as the property of its geometric realization (see [18, Section 1.1] for more details).

Definition 4.1. The reduced Euler characteristic $\tilde{\chi}(\Delta)$ of a simplicial complex Δ is defined as:

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim(\Delta)} (-1)^i f_i(\Delta),$$

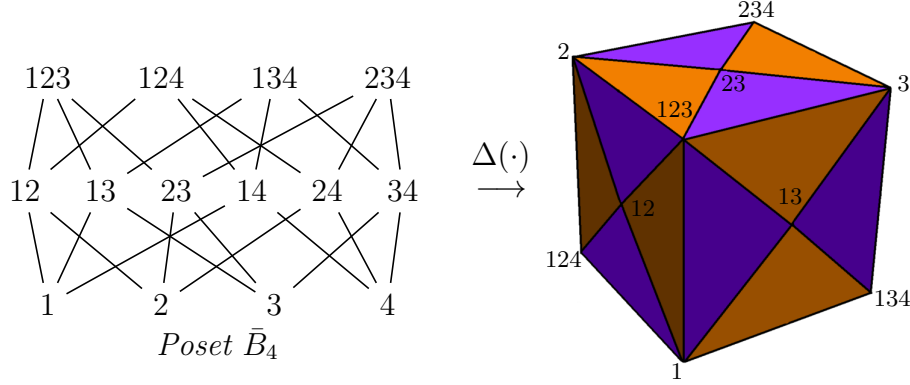
where $f_i(\Delta)$ is the number of i -faces of Δ .

Example 4.2. The order complexes of \bar{B}_3 and \bar{B}_4 :



$\Delta(\bar{B}_3)$ is a hexagon; it has the homotopy type of \mathbb{S}^1 .

$$\tilde{\chi}(\Delta(\bar{B}_3)) = -1 + 6 - 6 = -1.$$



$\Delta(\bar{B}_4)$ is a tetrakis hexahedron¹; it has the homotopy type of \mathbb{S}^2 .

$$\tilde{\chi}(\Delta(\bar{B}_4)) = -1 + 14 - 36 + 24 = 1.$$

Theorem 4.3 (Euler-Poincaré formula). *For a simplicial complex Δ ,*

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim(\Delta)} (-1)^i \tilde{\beta}_i(\Delta),$$

where $\tilde{\beta}_i(\Delta)$ is the reduced Betti number (i.e., dimension of i th reduced homology) of Δ .

For posets (co)homology, we refer to the survey [18, Section 1.5].

4.2. Möbius invariant of a coset poset.

Definition 4.4. *Let P be a poset. The Möbius function μ on P is defined recursively on the closed intervals of P as follows:*

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z) & \text{otherwise.} \end{cases}$$

If P is a bounded poset, then the Möbius invariant $\mu(P)$ of P is $\mu(\hat{0}, \hat{1})$.

Let P_1 and P_2 be posets with Möbius functions μ_1 and μ_2 . Then $\mu(P_1 \times P_2) = \mu_1(P_1) \times \mu_2(P_2)$ [18, Proposition 1.2.1].

Example 4.5. $\mu(B_n) = (-1)^n$, because $B_n = B_1 \times \dots \times B_1$ (n times), and $\mu(B_1) = -1$.

Proposition 4.6. [18, Proposition 1.2.6] *For any poset P ,*

$$\mu(\hat{P}) = \tilde{\chi}(\Delta(P)).$$

¹<https://commons.wikimedia.org/w/index.php?curid=40239574>

Lemma 4.7. *Let P be a poset and $x < y$. Then*

$$\mu(x, y) = - \sum_{x < z \leq y} \mu(z, y).$$

Proof. The result follows from Proposition 4.6 and the fact that $\Delta(P) = \Delta(P^*)$, with P^* the dual poset of P (i.e. order reversed). \square

Definition 4.8. *For a group G and its subgroup H , the coset poset $C(H, G)$ is defined to be the poset of (proper) right cosets Kg with $g \in G$ and $K \in [H, G]$, ordered by inclusion, and*

$$\hat{C}(H, G) := C(H, G) \sqcup \{\emptyset, G\}.$$

Lemma 4.9. *Let $g \in G$ and $H_1, H_2 \in [H, G]$. Then $[H_1, H_2]$ is poset isomorphic to $[H_1g, H_2g]$.*

Proof. The multiplication by g is an order preserving bijection between the posets $[H_1, H_2]$ and $[H_1g, H_2g]$. \square

Proposition 4.10 ([5], Section 3). *Let $[H, G]$ be an interval of finite groups. Then*

$$\mu(\hat{C}(H, G)) = - \sum_{H \leq K \leq G} \mu(K, G) |G : K|.$$

Proof. For a subgroup K of G , let R_K denote a set of coset representatives of K in G . Then using Lemma 4.7, we have

$$\begin{aligned} \mu(\hat{C}(H, G)) &= \mu(\emptyset, G) = - \sum_{\emptyset < Kg \leq G} \mu(Kg, G) \\ &= - \sum_{H \leq K \leq G} \sum_{g \in R_K} \mu(Kg, G) \\ &= - \sum_{H \leq K \leq G} \mu(K, G) |G : K|. \end{aligned}$$

The last equality follows from the fact that for a fixed $g \in G$, the interval $[Kg, G]$ is poset isomorphic to the interval $[K, G]$, and $|R_K| = |G : K|$. \square

Remark 4.11. *For $[H, G]$ boolean and $K \in [H, G]$, $\mu(K, G) = (-1)^{\ell(K, G)}$ and $(-1)^{\ell(K, G)} = (-1)^{\ell(H, G)} (-1)^{\ell(H, K)}$. Thus we get*

$$\mu(\hat{C}(H, G)) = -(-1)^{\ell(H, G)} \hat{\varphi}(H, G).$$

Lemma 4.12. *Let $[H, G]$ be a boolean interval. For $P = C(H, G)$, the nontrivial reduced Betti number of the order complex $\Delta(P)$ is exactly the dual Euler totient $\hat{\varphi}(H, G)$.*

Proof. By Proposition 4.6 and Theorem 4.28,

$$\mu(\hat{P}) = (-1)^{\ell(P)} \beta_{\ell(P)}(\Delta(P)),$$

but $\ell(P) = \ell(H, G) - 1$, so by Remark 4.11, $\beta_{\ell(P)}(\Delta(P)) = \hat{\varphi}(H, G)$. \square

Inspired by Remark 4.11, we extend the notion of dual Euler totient to any interval of finite groups $[H, G]$ as follows:

$$\hat{\varphi}(H, G) := -(-1)^{\ell(H, G)} \mu(\hat{C}(H, G)).$$

Thus we get,

Corollary 4.13. *For any interval of finite groups $[H, G]$,*

$$\hat{\varphi}(H, G) = (-1)^{\ell(H, G) - \ell(T, G)} \hat{\varphi}(T, G)$$

with $[T, G]$ the top interval of $[H, G]$.

Proof. Let $K \in [H, G] \setminus [T, G]$. It suffices to prove that $\mu(K, G) = 0$. Let

$$T(K) := \bigwedge_{R \in (K, G] \cap [T, G]} R.$$

Then $T(K)$ is the smallest element of $[T, G]$ containing K , so

$$(1) \quad (K, G] \cap [T, G] = [T(K), G].$$

Now,

$$-\mu(K, G) = \sum_{R \in (K, G] \setminus [T, G]} \mu(R, G) + \sum_{R \in (K, G] \cap [T, G]} \mu(R, G).$$

By induction on $|(K, G] \setminus [T, G]|$, we can assume the first component of the above sum to be zero. Now by definition

$$\mu(T(K), G) = - \sum_{R \in (T(K), G]} \mu(R, G),$$

so by (1), the second component of the above sum is also zero. \square

We can also extend the notion of Euler totient as follows.

$$\varphi(H, G) := \sum_{K \in [H, G]} \mu(K, G) |K : H|$$

Corollary 4.14. *For any interval of finite groups $[H, G]$,*

$$\varphi(H, G) = |T : H| \cdot \varphi(T, G)$$

with $[T, G]$ the top interval of $[H, G]$.

Proof. As for the proof of Proposition 4.13. \square

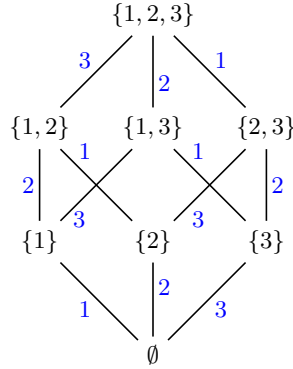
This matches with Remark 3.16.

4.3. Cohen-Macaulay posets and edge labeling. A finite poset P is pure if all the maximal chains $x_0 < x_1 < \cdots < x_r$ have the same length r . A finite poset P is graded if it is bounded and pure. A cover relation $x \lessdot y$ in a poset P is the relation $x < y$ such that $x \leq z < y$ implies $x = z$. The cover relations on P will be identified with the edges on its Hasse diagram. An edge labeling of a poset P is a map $\lambda : E(P) \rightarrow A$, where $E(P)$ is the set of edges and A is some poset (for our purpose A will be the set of integers).

Definition 4.15 ([18]). Let λ be an edge labeling of a graded poset P . λ is said to be an edge-lexicographical labeling (EL-labeling, for short) if for each closed interval $[x, y]$ of P ,

- (i) there is a unique strictly increasing maximal chain.
- (ii) this maximal chain is lexicographically first among all other maximal chains of $[x, y]$.

Example 4.16. Here is an EL-labeling for B_3 (it generalizes to B_n).



Definition 4.17. A dual EL-labeling on a graded poset P is an EL-labeling on its dual poset (i.e. order reversed).

Definition 4.18 ([1]). A graded poset P is called Cohen-Macaulay (over \mathbb{C}) if for any open interval (x, y) in P

$$\tilde{\beta}_i(\Delta((x, y)), \mathbb{C}) = 0 \text{ for all } i < \dim(\Delta((x, y))).$$

Theorem 4.19. If \hat{P} is a graded poset which admits an EL-labeling, then it is Cohen-Macaulay. Moreover, the order complex $\Delta(P)$ has the homotopy type of a wedge of spheres \mathbb{S}^d with $d = l(P)$. The number of spheres is one of the following equal quantities:

- (1) the number of (weakly) decreasing maximal chains in \hat{P} .
- (2) the Möbius invariant $\mu(\hat{P})$ times $(-1)^d$.
- (3) the reduced Betti number $\tilde{\beta}_d(\Delta(P))$.
- (4) the reduced Euler characteristic $\tilde{\chi}(\Delta(P))$ times $(-1)^d$.

Proof. Merge [1, Theorem 3.2] and [18, Theorem 3.2.4]. \square

Remark 4.20. *Note that a Cohen-Macaulay poset need not have an EL-labeling (see [1, p16]).*

Remark 4.21. *The extension of the EL-labeling of Example 4.16 to B_n implies that it is Cohen-Macaulay. Moreover, by Example 4.5, $\mu(B_n) = (-1)^n$, so by Theorem 4.19, $\Delta(\bar{B}_n)$ has the homotopy type of \mathbb{S}^{n-2} .*

4.4. An edge labeling for $\hat{C}(H, G)$. Let us fix some notations which shall be used in the sequel. The interval of finite groups $[H, G]$ will be boolean. Then the poset $\hat{C}(H, G)$ is graded. Let K_1, \dots, K_r be the atoms in the interval $[H, G]$. Then, the coatoms are of the form $M_i := K_i^c = \bigvee_{j \neq i} K_j$. We observe that:

Lemma 4.22. *Any cover relation of $\hat{C}(H, G)$ is of the form $\emptyset \triangleleft Hg$ or $Xg \triangleleft Yg$ with $g \in G$, $Y = X \vee K_i$ for some i and $K_i \not\subseteq X$.*

Definition 4.23. *Let the edge labeling el on $\hat{C}(H, G)$ be:*

- $el(\emptyset \triangleleft Hg) = 0$
- $el(Xg \triangleleft Yg) = \begin{cases} -i & \text{if } Xg = Yg \cap M_i, \\ +i & \text{otherwise} \end{cases}$

This edge labeling has been suggested by Russ Woodroffe [19].

Lemma 4.24. *Let $Y = X \vee K_i$ with $K_i \not\subseteq X$. Then $Xg = Yg \cap M_i$ iff $g \in M_i$.*

Proof. Suppose that $[H, G]$ is boolean of rank n . Let $\phi : [H, G] \rightarrow B_n$ be the lattice isomorphism such that $\phi(K_i) = \{i\}$. Then $\forall X \in [H, G]$,

$$K_i \not\subseteq X \Leftrightarrow \{i\} \not\subseteq \phi(X) \Leftrightarrow \phi(X) \subseteq \{i\}^c \Leftrightarrow X \subseteq K_i^c = M_i.$$

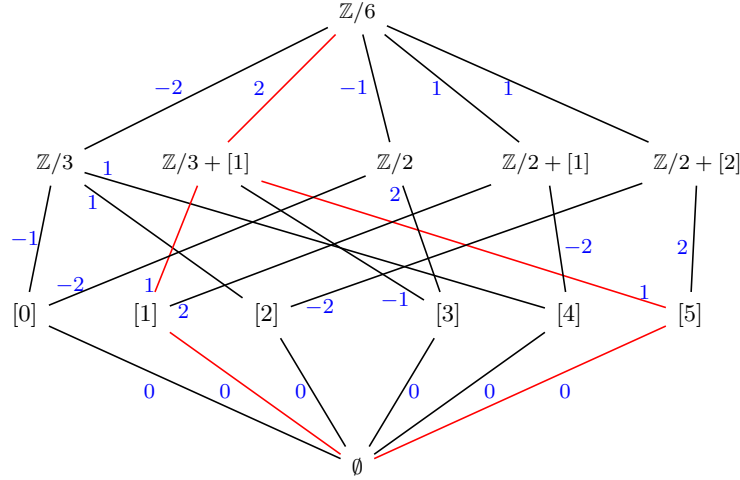
Now assume that $g \in M_i$, then

$$Y \wedge M_i = (X \vee K_i) \wedge M_i = (X \wedge M_i) \vee (K_i \wedge M_i) = X \vee H = X.$$

The second equality uses the distributivity of $[H, G]$ boolean, and the third follows from the fact that $X \subseteq M_i$ and $K_i \wedge M_i = K_i \wedge K_i^c = H$. This implies that $Xg = Yg \cap M_i g = Yg \cap M_i$ as $g \in M_i$.

Conversely if $Xg = Yg \cap M_i$, then $g \in Xg \subseteq M_i$. \square

Example 4.25. *The edge labeling, as in Definition 4.23, on $\hat{C}([0], \mathbb{Z}/6)$:*



Now we discuss a necessary and sufficient condition for the edge labeling el (see Definition 4.23) to be a dual EL-labeling. Each interval in $\hat{C}(H, G)$ should have a unique strictly increasing maximal chain (from top to bottom) which is also lexicographically first. We will consider two kind of intervals individually:

Case 1. $[H_1g, H_2g]$:

By the boolean structure of $[H, G]$, we can write H_2 uniquely as

$$H_1 \vee (\vee_{i \in I} K_i)$$

with $I \subseteq \{1, \dots, r\}$. So by Lemma 4.9, any maximal chain in $[H_1g, H_2g]$ is of the form

$$H_1g \leq (H_1 \vee K_{i_0})g \leq \dots \leq (H_1 \vee (\vee_{i \in I} K_i))g = H_2g.$$

The labeling of all such chains is same upto permutation, because the sign of a label i does not depend on choice of the chain, but depends only on the fact that $g \in M_i$ or not, by Lemma 4.24. By the boolean structure, every permutation occurs and so we can choose the lexicographically first which is also unique and strictly increasing.

Case 2. $[\emptyset, Kg]$:

Let $I = \{i_1, \dots, i_p\}$ such that $K = \vee_{i \in I} K_i$. Using the boolean structure of $[H, G]$, any maximal chain in $[\emptyset, Kg]$ is of the form

$$\emptyset \leq Hg' \leq (H \vee K_{\sigma(i_1)})g' \leq (H \vee K_{\sigma(i_1)} \vee K_{\sigma(i_2)})g' \leq \dots \leq (\vee_{i \in I} K_i)g' = Kg',$$

where $g' \in Kg$ and $\sigma \in S_p$.

Existence of a strictly increasing chain

- *Necessary condition:*

Since the label of the leftmost edge of the above chain is 0, for the

existence of a strictly increasing chain, it is necessary to find $g' \in Kg$ for which all the other labels are negative, which means that $g' \in \cap_{i \in I} M_i = K^{\mathbb{C}}$. So $[\emptyset, Kg]$ admits a strictly increasing maximal chain iff $K^{\mathbb{C}} \cap Kg \neq \emptyset$, iff $\exists k \in K$ and $\exists k' \in K^{\mathbb{C}}$ such that $k' = kg$ (i.e. $g = k^{-1}k'$), iff $g \in KK^{\mathbb{C}}$.

As this should hold $\forall g \in G$ and $\forall K \in [H, G]$, we conclude that $G = KK^{\mathbb{C}}$ and $[H, G]$ is group-complemented (see Definition 3.24).

- *Sufficient condition:*

The existence of $g' \in K^{\mathbb{C}} \cap Kg$ is sufficient because by Case 1, there exists a unique strictly increasing maximal chain in $[Hg', Kg']$ which by adding the last label 0, is still strictly increasing and lexicographically first on $[\emptyset, Kg]$.

Uniqueness of the strictly increasing chain

For the uniqueness, we just need to show that there exists a unique possible Hg' . Let $g_1, g_2 \in K^{\mathbb{C}} \cap Kg$, we have $g_1 = k_1g$, $g_2 = k_2g$ and hence $g_1g_2^{-1} = k_1k_2^{-1} \in K$. Moreover, $g_1g_2^{-1} \in K^{\mathbb{C}}$. Therefore $g_1g_2^{-1} \in K \cap K^{\mathbb{C}} = H$. Thus, $Hg_1 = Hg_2$.

From the above discussion, we conclude the following result.

Theorem 4.26. *Let $[H, G]$ be a boolean interval. The edge labeling el on $\hat{C}(H, G)$, is a dual EL-labeling iff $[H, G]$ is group-complemented.*

Remark 4.27. *The group-complemented assumption is necessary and sufficient for Case 2, but Case 1 works in general.*

For better understanding, we discuss an easy example. Since the interval $[[0], \mathbb{Z}/6]$ is group-complemented, the labeling of Example 4.25 is a dual EL-labeling by Theorem 4.26. Moreover, there are two maximal decreasing chains, so by Theorem 4.19, the order complex of the proper part of $\hat{C}([0], \mathbb{Z}/6)$ has the homotopy type of the wedge of two \mathbb{S}^1 .

Theorem 4.28. *Let $[H, G]$ be a boolean group-complemented interval. Then $\hat{C}(H, G)$ is Cohen-Macaulay.*

Proof. Since the order complex is invariant by dual, so is the Cohen-Macaulay property. Therefore the result follows from Theorem 4.26 and Theorem 4.19. \square

Theorem 4.29. *Let $[H, G]$ be a boolean group-complemented interval. For $P = C(H, G)$, the nontrivial reduced Betti number of the order complex $\Delta(P)$ is exactly the number of cosets Hg such that $\langle Hg \rangle = G$, which is nonzero.*

Proof. It follows from Lemmas 3.15, 3.29, 4.12 and Theorem 3.1. \square

Question 4.30. *Is $\hat{C}(H, G)$ Cohen-Macaulay for any boolean interval $[H, G]$? If so, is the nontrivial reduced Betti number nonzero?*

4.5. Examples. Apart from the group-complemented intervals of the previous section, we will exhibit other classes of examples for which Question 4.30 has a positive answer.

The following result was pointed out to us by Russ Woodroffe:

Theorem 4.31 ([1], p14). *Any graded poset of length one or two is Cohen-Macaulay. Any graded poset of length 3 is Cohen-Macaulay iff its proper part is connected.*

Proposition 4.32. *Let $[H, G]$ be a boolean interval of finite groups. Then the proper part of $\hat{C}(H, G)$ is connected.*

Proof. Let Hg and Hg' be two atoms of $\hat{C}(H, G)$. Let K_1, \dots, K_n be the atoms of $[H, G]$. Then by the boolean structure, $g'' = g'g^{-1}$ is a product of elements in some K_i , i.e.

$$g'' = k_{\tau(1)}k_{\tau(2)} \dots k_{\tau(s)}$$

with $\tau(i) \in \{1, 2, \dots, n\}$ and $k_{\tau(i)} \in K_{\tau(i)}$.

Now, $g' = g''g$, so we get that $g' = k_{\tau(1)}k_{\tau(2)} \dots k_{\tau(s)}g$. Let $g_i = k_{\tau(s-i)} \dots k_{\tau(s)}g$. Then Hg_i and Hg_{i+1} are connected via $K_{\tau(s-i-1)}g_i$. We deduce that Hg and Hg' are connected. But any element of the proper part of $\hat{C}(H, G)$ is connected to an atom. The result follows. \square

Lemma 4.33. *Let $[H, G]$ be a rank 2 boolean interval of finite groups. Then the dual Euler totient $\hat{\varphi}(H, G) \geq 1$.*

Proof. Let K_1, K_2 be the atoms of $[H, G]$. Then

$$\hat{\varphi}(H, G) = |G : H| - |G : K_1| - |G : K_2| + |G : G|.$$

But $|G : H| = |G : K_i| \cdot |K_i : H| \geq 2|G : K_i|$. It follows that $\hat{\varphi}(H, G) \geq |G : G| = 1$. \square

Corollary 4.34. *The graded coset poset of any rank 2 boolean interval $[H, G]$, is Cohen-Macaulay and the nontrivial reduced Betti number is nonzero.*

Proof. By Proposition 4.32, Lemma 4.33, Theorems 4.31 and 4.19. \square

Remark 4.35. *By GAP, at index $|G : H| < 32$, there are, up to equivalence, 612 boolean intervals (i.e. $1 + 241 + 337 + 33$, according to the rank 0, 1, 2, 3). They are all group-complemented except $[D_8, A_2(2)]$ and $[S_3, A_2(2)]$, both of rank 2.*

Corollary 4.36. *The graded coset poset of any boolean interval $[H, G]$ of index < 32 , is Cohen-Macaulay and the nontrivial reduced Betti number is nonzero.*

Proof. By Remark 4.35, Theorems 4.28, 4.29 and Corollary 4.34. \square

In the following example, for the notions of BN-pair, Coxeter system, Borel subgroup, spherical building, simple groups of Lie type, Chevalley groups and Dynkin diagram, we refer to the books [3, 4, 6, 7].

Example 4.37. *Let G be a finite group with a BN-pair, let B be the corresponding Borel subgroup and (W, S) the Coxeter system. Let $n := |S|$ be the rank of the BN-pair. Then the interval $[B, G]$ is boolean of rank n [6, Theorem 8.3.4]. The order complex of $C(B, G)$ is equivalent to the spherical building associated to the BN-pair [7, Section 5.7] as abstract simplicial complexes. It is Cohen-Macaulay ([2], [4, Remark 3 p94]) and has the homotopy type of a wedge of $r(\geq 1)$ spheres \mathbb{S}^{n-1} [4, Theorem 2 p93]. It follows that the dual Euler totient $\hat{\varphi}(B, G) = r \neq 0$, so by Theorem 3.21, the boolean interval $[B, G]$ is linearly primitive. Any finite simple group G of Lie type (over a finite field of characteristic p) admits a BN-pair (except Tits group) and r is the p -contribution in the order of G [17, Section 4, (ii')]. If moreover, G is a Chevalley group, then n is the number of vertices in its Dynkin diagram.*

The rank 2 boolean intervals $[D_8, A_2(2)]$ and $[S_3, A_2(2)]$ of Remark 4.35, have dual Euler totients $\hat{\varphi} = 2^3$ and 15, respectively. The first comes from a BN-pair, but not the second.

Remark 4.38. *By GAP and the help of Alexander Hulpke, there are four boolean intervals $[H, G]$ of rank 3, with G simple and $|G| \leq 4 \cdot 10^6$ (given by $G = A_3(2)$, $C_3(2)$, ${}^2A_2(5^2)$ and ${}^2A_3(3^2)$; all of Lie type). None of them is group-complemented. Their corresponding dual Euler totients $\hat{\varphi}$ are 2^6 , 2^9 , 3899 and 3968, respectively. The first two come from BN-pairs, but not the two last (because 3899 and 3968 are not prime-powers). Using SageMath [14], we can check that the coset poset of the third is also Cohen-Macaulay (we don't know about the last one).*

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